MATH 245 S25, Exam 1 Solutions

1. Carefully define the following terms: tautology, xor.

   A tautology is a proposition that is logically equivalent to T (or, that is always T). Xor is a propositional operator denoted $\oplus$, where proposition $p \oplus q$ is F if p, q are either both T or both F (and T otherwise). Alternate solution: Given any propositions p, q, the proposition $p \oplus q$, p xor q, is T when exactly one of p, q are T, and F otherwise.

2. Carefully state the following theorems: Distributivity (for Propositions), Simplification Semantic Theorem.

   The distributivity semantic theorem says, for all propositions p, q, r, that $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$ and also $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$. The simplification semantic theorem says, for any propositions p, q, that $p \wedge q \vdash p$.

3. Let p, q, r, s be propositions. Prove that $(p \wedge q) \to (r \wedge s)$ is NOT equivalent to $(r \wedge s) \to (p \wedge q)$.

   God help anyone who built a 16×8 truth table, what a huge waste of time. All that is needed is one (complete) specific single row of this truth table, where the two propositions disagree. We need to make $p \wedge q$ false and $r \wedge s$ true (or the other way around), and there are lots of ways to do this. One way: take p to be F and q, r, s all T. Now $p \wedge q$ is F and $r \wedge s$ is T, so $(p \wedge q) \to (r \wedge s)$ is T but $(r \wedge s) \to (p \wedge q)$ is F.

4. For arbitrary $n \in \mathbb{N}_0$, calculate and simplify $\frac{(3n)!}{(3(n+1))!}$.

   Note that $(3(n+1))! = (3n+3)! = (3n+2)!(3n+3) = (3n+1)!(3n+2)(3n+3) = (3n)!(3n+1)(3n+2)(3n+3)$, using the definition of factorial three times. Plugging in, we get $\frac{(3n)!}{(3(n+1))!} = \frac{(3n)!}{(3n)!(3n+1)(3n+2)(3n+3)} = \frac{1}{(3n+1)(3n+2)(3n+3)}$. If you really want to play walking calculator, you can optionally multiply this out to get $\frac{1}{27n^3+54n^2+33n+6}$.

5. Carefully state the commutativity theorem for disjunction. Then, prove it without truth tables, using at most two cases.

   Statement: For any propositions p, q, we have $p \vee q \equiv q \vee p$.

   PROOF 1: Two cases: either p, q are both F or not.
   If p, q ARE both F, then $p \vee q$ is F, but also q, p are both F so $q \vee p$ is F. If p, q are NOT both F, then $p \vee q$ is T but also q, p are not both F so $q \vee p$ is T. In both cases $p \vee q$ and $q \vee p$ agree.

   PROOF 2: Two cases: either $p \vee q$ is T or F. If T, then at least one of p, q is T. But then also $q \vee p$ is T. If instead $p \vee q$ is F, then both of p, q are F. But then also $q \vee p$ is F. In both cases $p \vee q$ and $q \vee p$ agree.

6. Without truth tables, prove that, for all propositions p, q, r, we have $p \to q, q \to r, r \to p \vdash p \leftrightarrow q$.

We must begin by letting p, q, r be arbitrary propositions, and assuming that $p \to q, q \to r, r \to p$ are all T.

Next, we must prove $q \to p$. There are many correct ways to do this, here are three of them.
METHOD 1: Direct proof. Suppose that q is true. By modus ponens with $q \to r$, we conclude r is true. By modus ponens again with $r \to p$, we conclude p is true. Hence $q \to p$.
METHOD 2: Cases on q. If q is true, then by modus ponens r is true, and by modus ponens again p is true. So by addition $p \vee (\neg q)$ is true. If instead q is false, then by addition $p \vee (\neg q)$ is true. In both cases $p \vee (\neg q)$ is true, so by conditional interpretation $q \to p$ is true.
METHOD 3: Since $r \to p$ is true, by conditional interpretation $p \vee (\neg r)$ is true. Now we do cases. If p is true, then by addition $p \vee (\neg q)$ is true. If instead $\neg r$ is true, then by modus tollens with $q \to r$, $\neg q$ is true. By addition $p \vee (\neg q)$ is true. In both cases $p \vee (\neg q)$ is true, so by conditional interpretation $q \to p$ is true.

Correct proofs should end with Thm 2.17(b), although it's fine to call it a "theorem in the book". This allows us to combine the hypothesis $p \to q$ with the recently proved $q \to p$ to get $p \leftrightarrow q$ (which must be the final statement).

7. Let $x \in \mathbb{Z}$. Prove that if x^3 is odd, then x is odd.

A direct proof is very difficult, not recommended. Instead we use a contrapositive proof. Suppose that x is not odd. Then, by Cor 1.8 (although it's fine to call it a "theorem in the book"), x is even. Hence there is some integer y so that $x = 2y$. We now calculate $x^3 = (2y)^3 = 8y^3 = 2(4y^3)$. Since $4y^3$ is an integer, x^3 is even. By Cor 1.8 AGAIN, we conclude that x^3 is not odd.

8. Prove or disprove: $\forall a, b, c \in \mathbb{Z}$, if $a|c$ and $b|c$, then $ab|c$.

The statement is false, and needs a counterexample. That is, $\neg \forall a, b, c\ (p \wedge q) \to r \equiv \exists a, b, c\ (p \wedge q) \wedge \neg r$.

Many solutions are possible. Take $a = 2, b = 4, c = 12$. Now $a|c$ since $2 \cdot 6 = 12$ and $b|c$ since $4 \cdot 3 = 12$. We now prove $ab \nmid c$ by contradiction. Suppose instead, by way of contradiction, that $ab|c$. Then there would be an integer k satisfying $abk = c$, i.e. $2 \cdot 4 \cdot k = 12$, so $k = \frac{12}{8} = 1.5$. Since 1.5 isn't an integer, we have our contradiction.

9. Prove or disprove: $\forall x \in \mathbb{Z},\ |7x - 10| \geq 2$.

The statement is true. We begin by letting $x \in \mathbb{Z}$ be arbitrary. We have two cases, motivated by the way the absolute value is calculated.
Case $x \geq 2$: Multiplying by 7, we get $7x \geq 14$, so $7x - 10 \geq 14 - 10 = 4$. In particular $7x - 10 > 0$ so $|7x - 10| = 7x - 10$, but also $|7x - 10| = 7x - 10 \geq 4 \geq 2$.
Case $x \leq 1$ (i.e. $x < 2$): Multiplying by 7, we get $7x \leq 7$, so $7x - 10 \leq 7 - 10 = -3$. Multiplying by -1 we get $-(7x - 10) \geq 3$. In particular $7x - 10 < 0$ so $|7x - 10| = -(7x - 10) \geq 3 \geq 2$.

In both cases $|7x - 10| \geq 2$.

10. Prove or disprove: $\forall x \in \mathbb{Z}, \forall y \in \mathbb{Z},\ |7x - 10y| \geq 2$.

The statement is false, and needs a specific counterexample. Many choices are possible.
SOLUTION 1: Take $x = y = 0$, now $|7x - 10y| = |0 - 0| = |0| = 0 \not\geq 2$.
SOLUTION 2: Take $x = 10, y = 7$, now $|7x - 10y| = |70 - 70| = |0| = 0 \not\geq 2$.
SOLUTION 3: Take $x = 3, y = 2$, now $|7x - 10y| = |21 - 20| = |1| = 1 \not\geq 2$.